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An Introduction to Neutrix Composition of Distributions and Delta Function

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ABSTRACT

The composition of the distribution $\delta^{(s)}(x)$ and an infinitely differentiable function *f(x)* having a simple zero at the point $x = x_0$ is defined by Gel'fand Shilov by the equation $\delta^{(s)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{f'(x)}\frac{1}{dx}\right]^s \delta(x - x_0)$. It is shown how this definition can be extended to functions *f(x)* which are not necessarily infinitely differentiable or not having simple zeros at the point $x = x_0$, by defining $\delta^{(s)}(f(x))$ as the limit or neutrix limit of the sequence $\{\delta_n^{(s)}(f(x))\}$, where $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. A number of examples are given.

Keywords: Distribution, delta-function, composition of distributions, neutrix, neutrix limit.

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INTRODUCTION

The Dirac delta function $\delta(x)$ has a long history. Its first appearance seems to have been in Fourier's *Theorie Analytique de la Chaleur*. Kirchoff (1882) later defined $\delta(x)$ by

$$\delta(x) = \lim_{\mu\to\infty} \pi^{-1/2}\mu \exp(-\mu^2 x^2)\,.$$

Clearly $\delta(x)=0$ for $x\neq 0$ and $\delta(0)=\infty$. He also defined

$$\int_{-\infty}^{x}\delta(t)dt=\lim_{\mu\to\infty}\pi^{-1/2}\mu\int_{-\infty}^{x}\exp(-\mu^2t^2)dt$$

this implies that

$$\int_{-\infty}^{x}\delta(t)dt=\begin{cases}0, & x<0\\ 1 & x>0.\end{cases}$$

In this sense, it is apparent that the delta function is not a function in the normal sense. Later Dirac treated the delta function as the function which is everywhere equal to zero except at the origin where it is infinite, in such a sense that it satisfies

$$\int_{-\infty}^{\infty}\delta(t)dt=1$$

Further Dirac thought of δ as a unit point charge at the origin. Moreover, he thought of δ' the derivative of δ as a dipole of unit electric moment at the origin since

$$\int_{-\infty}^{\infty}x\delta'(x)dx=\lim_{\mu\to\infty}\pi^{-1/2}\mu\int_{-\infty}^{\infty}x\left[\exp(-\mu^2x^2)\right]'dx=-1.$$

The delta function was next used by Heaviside (1893). Heaviside's function H is the locally summable function defined to be equal to 0 for $x<0$ and equal to 1 for $x>0$. Heaviside also appreciated that the derivative of H was in some sense equal to δ.

Similarly, higher derivatives of δ can be used to represent more complicated multiple-layers and have been used in the physical and engineering science for some time, (see Kilicman (2001), Kilicman (1999) and Kilicman (2004)). However, several researchers consider the Dirac delta function as a singular distribution. In this study we extend the definition of composition to $\delta^{(s)}(f(x))$ where the functions $f(x)$ are not necessarily infinitely differentiable or not having simple zeros.

In the following, we let D be the space of infinitely differentiable functions φ with compact support and let $D[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$. A *distribution* is a continuous linear functional defined on D. The set of all distributions defined on D is denoted by D' and the set of all distributions defined on $D[a,b]$ is denoted by $D'[a,b]$. Two distributions F and G are equal if and only if $\langle F,\varphi\rangle = \langle G,\varphi\rangle$ for all φ in D. If f is a summable function, it defines a distribution, also denoted by f, by defining $\langle f,\varphi\rangle$, its value at φ as

$$\langle f,\varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)dx.$$

If f is a differentiable function, it follows easily that

$$\langle f',\varphi\rangle = -\langle f,\varphi'\rangle \tag{1}$$

and so if f is not differentiable in the normal sense, we define its derivative f' by equation (1). In general we have

$$\langle f^{(r)},\varphi\rangle = (-)^r \langle f,\varphi^{(r)}\rangle.$$

As an example, let H denote Heaviside's function. Then

$$\langle H,\varphi\rangle = \int_0^{\infty} \varphi(x)dx$$

and so

$$\langle H',\varphi\rangle = -\langle H,\varphi'\rangle = \int_0^{\infty} \varphi'(x)dx = \varphi(0)\,.$$

We denote H' by δ, Dirac's delta function so that $\langle \delta,\varphi\rangle = \varphi(0)$ and in general

$$\langle \delta^{(r)},\varphi\rangle = (-)^r \varphi^{(r)}(0)$$

for $r = 0, 1, 2\ldots.$

We define the locally summable functions x_+^λ, x_-^λ, $|x|^\lambda$ and $\operatorname{sgn} x|x|^\lambda$ for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0 & x < 0, \end{cases} \qquad x_-^\lambda = \begin{cases} |x|^\lambda, & x < 0, \\ 0, & x > 0, \end{cases}$$

$$|x|^\lambda = x_+^\lambda + x_-^\lambda, \qquad \operatorname{sgn} x|x|^\lambda = x_+^\lambda - x_-^\lambda.$$

The distributions x_+^λ and x_-^λ are then defined inductively for $\lambda < -1$ and $\lambda \neq -2, -3, \ldots$ by

$$\left(x_+^\lambda\right)' = \lambda x_+^{\lambda-1} \quad \left(x_-^\lambda\right)' = -\lambda x_-^{\lambda-1}$$

and the distributions |*x*|_ and sgn *x*|*x*|_ are then defined by

$$|x|^\lambda = x_+^\lambda + x_-^\lambda, \qquad \operatorname{sgn} x|x|^\lambda = x_+^\lambda - x_-^\lambda.$$

The distribution x^{-1} is defined by $\left(\ln|x|\right)'$.

Now let $\rho(x)$ be a function in D having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots,$ it follows that

$$\lim_{n\to\infty} \langle \delta_n(t), \varphi(t) \rangle = \lim_{n\to\infty} \int_{-1/n}^{1/n} \delta_n(t)\varphi(t)dt = \langle \delta, \varphi \rangle = \varphi(0),$$

for arbitrary φ in D and so we see that $\{\delta_n(x)\}$ is a sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. More generally, $\{\delta_n^{(r)}(x)\}$ is a sequence of infinitely differentiable functions converging to $\delta^{(r)}(x)$.

If F is a distribution, it is the r-th derivative, for some r, of a summable function f on a bounded interval (a, b). We can therefore define the convolution $\left(F * \delta_n\right)(x) = F_n(x)$ by

$$\left(F * \delta_n\right)(x) = \left\langle F(x-t), \delta_n(t) \right\rangle = \left\langle f(x-t), \delta_n^{(r)}(t) \right\rangle = \int_{-1/n}^{1/n} f(x-t)\delta_n^{(r)}(t)dt$$

on the interval (a, b). It follows that $\{F_n(x)\}$ is a sequence of infinitely differentiable functions converging to $F(x)$ on the interval (a, b).

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta(f(x))$ by the equation

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$$

(see Gel'fand and Shilov (1964)).

Note that some certain divergent integrals can be interpreted as distributions, see Eltayeb *et al.* (2010) and Kilicman and Eltayeb (2009). Then it is a difficult task to give a meaning to the expression $F(f)$ where F and f are singular distributions.

If $f'(x) > 0,$ then putting $t = f(x)$ and $\psi(x) = f'(x)\varphi(f(x)),$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} \delta_n(t)\varphi(t)dt &= \int_{-\infty}^{\infty} \delta_n(f(x))f'(x)\varphi(f(x))dx \\ &= \int_{-\infty}^{\infty} \delta_n(f(x))\psi(x)dx = \left\langle \delta_n(f(x)), \psi(x) \right\rangle\end{aligned}$$

and so

$$\begin{aligned}\lim_{n\to\infty}\int_{-\infty}^{\infty} \delta_n(f(x))\psi(x)dx = \varphi(0) &= \frac{\psi(x_0)}{f'(x_0)} \\ &= \frac{1}{f'(x_0)}\left\langle \delta(x - x_0), \psi(x) \right\rangle \\ &= \left\langle \delta(f(x)), \psi(x) \right\rangle.\end{aligned}$$

If $f'(x) < 0$, we would have

$$\langle \delta(f(x)), \psi(x) \rangle = \frac{1}{|f'(x_0)|} \langle \delta(x - x_0), \psi(x) \rangle$$

and so

$$\delta(f(x)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

in either case. We have therefore proved that if $f(x)$ is an infinitely differentiable function having a single simple root at the point $x = x_0$, then $\delta(f(x))$ is the limit of the sequence $\{\delta_n(f(x))\}$. This suggests that if f is any function, we define $\delta(f(x))$ as the limit of the sequence $\{\delta_n(f(x))\}$, provided the limit exists.

MAIN RESULTS

Theorem 1. The composition $\delta(|x|)$ exists and $\delta(|x|) = \delta(x)$.

Proof. We have $\delta_n(|x|) = \delta_n(x)$, since δ_n is an even function, and so

$$\lim_{n\to\infty} \delta_n(|x|) = \lim_{n\to\infty} \delta_n(x) = \delta(x).$$

The function $|x|$ is not infinitely differentiable at the origin and so $\delta_n(|x|)$ is not defined by Gel'fand and Shilov's definition.

Theorem 2. The composition $\delta\left(|x|^{1/r}\right)$ exists and

$$\delta\left(|x|^{1/r}\right) = 0,$$

for $r = 2, 3, \ldots$.

Proof. We have

$$\left\langle \delta_n\left(|x|^{1/r} \right), \varphi(x) \right\rangle = \int_{-\infty}^{\infty} \delta_n\left(|x|^{1/r} \right) \varphi(x)dx$$

$$= n^{-r-1} \int_{-1}^{1} \rho(|t|)\varphi(n^{-r}t^r)dt$$

and it follows that

$$\lim_{n\to\infty} \left\langle \delta_n\left(|x|^{1/r} \right), \varphi(x) \right\rangle = 0$$

and so $\delta_n\left(|x|^{1/r} \right) = 0$ for $r = 2,3,....$

In order to give a more general definition for the composition of distributions, the following definition was given in Fisher (1985) and was originally called the composition of distributions.

Definition 1. *Let f be an infinitely differentiable function. We say that the distribution* $\delta^{(r)}(f(x))$ *exists and is equal to h on the open interval* (a, b) *if*

$$N - \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n^{(r)}(f(x))\varphi(x)dx = \langle h(x), \varphi(x) \rangle$$

for all φ in D[*a, b*]*, where N is the neutrix, see Van Der Corput (1959), having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions*

$$n^{\lambda} \ln^{r-1} n, \ln^r n : \quad \lambda > 0, r = 1,2,...$$

and all functions which converge to zero in the usual sense as n *tends to infinity.*

Note that taking the neutrix limit of a function *f*(*n*), is equivalent to taking the usual limit of Hadamard's finite part of *f*(*n*).

The following theorems were proved in Fisher (1983) and Fisher (1985) respectively.

Theorem 3. *The distribution* $\delta^{(r)}(x^s)$ *exists and*

$$\begin{aligned}\delta^{(r)}\left(x^{2s}\right) &= 0 \\ \delta^{(r)}\left(x^{2s-1}\right) &= \frac{r!}{s(rs+s-1)!}\delta^{(rs+s-1)}(x)\end{aligned}$$

for $r = 0,1,2,\ldots$ and $s = 1,2,\ldots$

Theorem 4. *The neutrix composition* $\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right)$ *exists and*

$$\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right) = 0$$

for $s = 0,1,2,\ldots$ and $(s+1)\lambda = 1,3,\ldots$ and

$$\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right) = \frac{(-1)^{(s+1)(\lambda+1)}s!}{\lambda[(s+1)\lambda-1]!}\delta^{(s+1)\lambda-1}(x)$$

for $s = 0,1,2,\ldots$ and $(s+1)\lambda = 2,4,\ldots$.

Theorem 5. The neutrix composition $\delta(x_+)$ exists and

$$\delta(x_+) = \frac{1}{2}\delta(x)\,.$$

Proof. We have

$$\begin{aligned}\left\langle \delta_n(x_+), \varphi(x)\right\rangle &= \int_{-1/n}^{1/n} \delta_n(x_+)\varphi(x)dx \\ &= \int_0^{1/n} \delta_n(x)\varphi(x)dx + \int_{-1/n}^{0} \delta_n(0)\varphi(x)dx \\ &= \int_0^{1/n} \delta_n(x)\varphi(x)dx + n\int_{-1}^{0} \rho(0)\varphi(x)dx\end{aligned}$$

it follows that

$$N-\lim_{n\to\infty}\left\langle \delta_n\left(x_+\right),\varphi(x)\right\rangle = \lim_{n\to\infty}\int_0^{1/n}\delta_n\left(x\right)\varphi(x)dx = \frac{1}{2}\left\langle \delta(x),\varphi(x)\right\rangle$$

and so $\delta(x_+) = \frac{1}{2}\delta(x)$.

For our next more substantial result, we need the following lemma, which can be easily proved by induction:

Lemma 1.

$$\int_{-1}^{1} t^i \rho^{(s)}(t)dt = \begin{cases} 0, & 0\le i<s \\ (-1)^s s!, & 1=s \end{cases}$$

and

$$\int_0^1 t^s \rho^{(s)}(t)dt = \frac{1}{2}(-1)^s s!$$

for $r = 0,1,2,\ldots$.

We now prove the following theorem.

Theorem 6. The neutrix composition $\delta^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]$ exists and

$$\delta^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right] = \sum_{k=0}^{s-1}\frac{\left[1+(-1)^k\right](2s)!(2s+1)!}{k!(2k+2)!}\delta^{(k)}(x), \qquad (2)$$

for $s = 1,2,\ldots$

Proof: This time we need to evaluate

$$N-\lim_{n\to\infty}\left\langle \delta_n^{(2s)}\left[\;|x|^{1/2}/\left(1+|x|^{1/2}\right)\right],\varphi(x)\right\rangle,$$

where $\varphi(x)$ is an arbitrary function $D[-1,1]$.

By Taylor's Theorem we have

$$\varphi(x)=\sum_{k=0}^{m-1}\frac{x^k\varphi^{(k)}(0)}{k!}+\frac{x^m\varphi^{(m)}(\zeta x)}{m!}$$

where $0<\zeta<1$. In order to evaluate equation (2), we have to evaluate

$$\begin{aligned}
&N-\lim_{n\to\infty}\left\langle \delta_n^{(2s)}\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right],\varphi(x)\right\rangle\\
&=N-\lim_{n\to\infty}\sum_{k=0}^{m-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{1}x^k\delta_n^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]dx\\
&\quad+N-\lim_{n\to\infty}\int_{-1}^{1}\frac{x^m}{m!}\delta_n^{(2s)}\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]\varphi^{(m)}(\zeta x)dx\\
&=N-\lim_{n\to\infty}\sum_{k=0}^{m-1}\frac{\left[1+(-1)^k\right]\varphi^{(k)}(0)}{k!}\int_{0}^{1}x^k\delta_n^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]dx\\
&\quad+N-\lim_{n\to\infty}\int_{-1}^{1}\frac{x^m}{m!}\delta_n^{(2s)}\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]\varphi^{(m)}(\zeta x)dx. \qquad (3)
\end{aligned}$$

Making substitution $nx^{1/2}/\left(1+x^{1/2}\right)=t,$ we have

$$dx=\frac{2tdt}{n\left(1-t/n\right)^3}.$$

Then for $n>1,$ we have

$$\begin{aligned}
\int_0^1 x^k\delta_n^{(2s)}\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]dx&=2n^{2s-2k-1}\int_0^1\frac{t^{2k+1}}{\left(1-t/n\right)^{2k+3}}\rho^{(2s)}(t)dt\\
&=2\sum_{i=0}^{\infty}\int_0^1\frac{(2k+i+2)!}{(2k+2)!}\frac{t^{2k+i+1}}{n^{-2s+2k+i+1}}\rho^{(2s)}(t)dt
\end{aligned}$$

and so

$$N-\lim_{n\to\infty}\int_0^1 x^k\delta_n^{(2s)}\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]dx=\frac{(2s)!(2s+1)!}{(2k+2)!}, \qquad (4)$$

and on using lemma, for $k = 0,1,2,...$ and $s = 1,2,...$

Note that in the particular case $s = 0,$ we have

$$\lim_{n\to\infty}\int_0^1 x^k \delta_n^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]dx = 2n^{-2k-1}\int_0^1 \frac{t^{2k+1}}{(1-t/n)^{2k+3}}\rho(t)dt, \qquad (5)$$

for $k = 0,1,2,...$

Next, we have

$$\int_{-1}^1 \left|x^m \delta_n^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]\right|dx \le 2n^{2s-2m-1}\int_{-1}^1\left|\frac{t^{2m+1}}{(1-t/n)^{2m+3}}\rho^{(2s)}(t)\right|dt$$
$$= O\left(n^{2s-2m-1}\right)$$

and so

$$\lim_{n\to\infty}\int_{-1}^1 \left|x^m \delta_n^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]\right|dx = 0,$$

for $s = 0,1,2,....$

Thus if $\varphi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_{-1}^1 \left|x^s \delta_n^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]\right|\varphi(x)dx = \quad 0. \qquad (6)$$

It now follows from equations (3), (4) and (6) with $m = s,$ that

$$N-\lim_{n\to\infty}\left\langle x^s \delta_n^{(2s)}\left[|x|^{1/2}/\left(1+|x|^{1/2}\right)\right],\varphi(x)\right\rangle = \sum_{k=0}^{s-1}\frac{\left[1+(-1)^k\right](2s)!(2s+1)!}{k!(2k+2)!}\varphi^{(k)}(0)$$
$$= \sum_{k=0}^{s-1}\frac{\left[1+(-1)^k\right](2s)!(2s+1)!}{k!(2k+2)!}\left\langle \delta^{(k)}(x),\varphi(x)\right\rangle,$$

and so the neutrix composition $\delta^{(2s)}\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]$ exists and

$$\delta^{(2s)}\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]=\sum_{k=0}^{s-1}\frac{\left[1+(-1)^k\right](2s)!(2s+1)!}{k!(2k+2)!}\delta^{(k)}(x),$$

for $s=1,2,\ldots.$ Note that in the particular case s = 0, it follows from equations (5) and (6) that the composition $\delta\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]$ exists and $\delta\left[\,|x|^{1/2}/\left(1+|x|^{1/2}\right)\right]=0.$

For further results on the neutrix composition of distributions, see Fisher *et al.* (2010).

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